On δ-shock model with a change point in intershock time distribution

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Abstract

In this paper, we study the reliability of a system that works under δ-shock model. That is, the system failure occurs when the time between two successive shocks is less than a given threshold δ. In a traditional setup of the δ shock model, the intershock times are assumed to have the same distribution. In the present setup, a change occurs in the distribution of the intershock times due to an environmental effect. Thus, the distribution of the intershock times changes after a random number of shocks. The reliability of the system is studied under this change point setup.

Keywords: Change point; Reliability; Shock model; MTTF; Discrete phase-type distribution.

1. Introduction

The lifetime properties of a system that operates under δ shock model have been earlier studied under the classical assumption that the shocks occur according to a Poisson process, i.e. when the intershock times have exponential distribution (see, e.g. Li et al. (1999), Li and Kong (2007)). Afterwards, the reliability of the corresponding system has been evaluated under various extensions and generalizations (Eryilmaz (2012), Parvardeh and Balakrishnan (2015), Eryilmaz (2017), Wang and Peng (2017)). Recent discussions on the topic are in Lorvand et al. (2020), Zhao et al. (2021), Goyal et al. (2021), Bohlooli-Zefreh et al. (2021), Goyal et al. (2022), Chadjiconstantinidis and Eryilmaz (2022), Ye et al. (2023).

So far in the literature, the δ-shock model has been studied only for the case when the interarrival times between shocks are identically distributed with a common distribution. In this paper, we aim to study the δ-shock model under the assumption that the interarrival times are independent but nonidentically distributed. Tuncel and Eryilmaz (2018) studied the survival function and the mean time to failure of the system under δ shock model when the times between successive shocks follow proportional hazard rate model assuming that the intershock times are nonidentical. In the present paper, we consider a particular model that makes the intershock times nonidentical. We consider the case when the law of intershock times changes after a point that follows a known probability distribution. Such a change may occur due to an unexpected environmental effect. Indeed, the study of change point in interarrival time distributions is of interest in inventory, statistical quality control, and queueing theory (see, e.g. Jain (2001)). Although the change point for the magnitudes of shocks has been considered under shock models (see, e.g. Eryilmaz and Kan (2019), Chadjiconstantinidis et al. 2022), to the best

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of our knowledge the change in the intershock time distribution has not been considered in the context of δ shock model. The present model is quite flexible since it allows to model stochastically increasing or decreasing interarrival times. Indeed, in some cases there might be an increase/decrease in times between successive shocks. In some other practical applications, the times between shocks follow a certain probabilistic law and there might be a change point which affects the distribution of the interarrival times.

The paper is organized as follows. In Section 2, the description of the system model is presented, and reliability characteristics such as survival function and mean time to failure of the system are obtained. Section 3 contains a numerical example on computation of the system mean time to failure.

2. The model with a random change point

Assume that the system is subjected to external shocks that arrive according to a process $N(t)$ defined by $N(t) = \sup\{n : S_n \le t\}$, where $S_n = \sum_{i=1}^{n} X_i$, $n \ge 1$, is the time of the occurrence of the n-th shock, X_1 is the time until occurrence of the first shock and X_i, $i \ge 2$ is the interarrival time between the $(i-1)$-th and i-th shock. The random variables $\{X_i, i \ge 1\}$ are assumed to be independent but not identically distributed.

Let N_δ denote the number of shocks until the failure of the system. Assume that for $i = 1, 2, \cdots, M$, the X_i's are independent and identically distributed (iid) random variables having a common distribution function (df) F_1, and for $i = M+1, M+2, \cdots,$ X_i's are iid random variables with a common df F_2. Let $p_\iota = \overline{F}_\iota(\delta) = 1 - F_\iota(\delta)$, $i = 1, 2$. It is assumed that M is a random variable and follows a particular distribution. The following result is immediate from the definition of the random variable N_δ. See also equation (2) of Eryilmaz and Kan (2019).

Lemma 1. *If X_i have a joint distribution F_1 for $i = 1, 2, \cdots, M$, and X_i have a joint distribution F_2 for $i = M+1, M+2, \cdots$, then*

$$\Pr(N_\delta = n \,|M = m\} = (1-p_2)p_1^m p_2^{n-m-1}, \tag{1}$$

for $n > m$, and

$$\Pr(N_\delta = n \,|M = m\} = (1-p_1)p_1^{n-1}, \tag{2}$$

for $n \le m$. ∎

The unconditional pmf of N_δ can be computed from by conditioning on M and using (1) and (2).

$$\begin{aligned}\Pr(N_\delta = n) &= \textstyle\sum_{m=1}^{\infty} \Pr(N_\delta = n \,|M = m\}\Pr(M = m) \\ &= (1-p_2)\textstyle\sum_{m=1}^{n-1} p_1^m\, p_2^{n-m-1} q_m + (1-p_1)p_1^{n-1} \sum_{m=n}^{\infty} q_m. \end{aligned} \tag{3}$$

The distribution of the random variable M plays an important role in studying the distribution of N_δ and hence the distribution of the system's lifetime. Recall that the random variable M describes the distribution of the change point and hence it should be modeled as a integer valued random variable. A proper waiting time distribution might be used to model the random variable M. The well-known waiting time distributions such as discrete phase-type distributions, which can be seen as the distributions of the time to absorption in an absorbing Markov chain, can be used to model the random variable M. The class of discrete phase-type distributions has been found to be suitable for modeling times between events such as shocks, claims and repairs (Neuts and Meier, 1981). To be more precisely, consider a finite discrete time Markov chain with state space $\{1, 2, \cdots, c, c+1\}$, where the first c states are the transient states and the last state $c+1$ is the absorbing state, and transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} \mathbf{Q} & \boldsymbol{u}^T \\ \mathbf{0} & 1 \end{pmatrix}_{(c+1)\times(c+1)},$$

where $\boldsymbol{u}^T$ is an $c \times 1$ column vector of nonnegative elements, $\mathbf{0} = (0, 0, \cdots, 0)$ is the $1 \times c$ row vector of zeroes (the superscript "T" denotes transpose), and $\mathbf{Q}$ is an $c \times c$ matrix with nonnegative elements. The probability distribution of the initial state is denoted by the $1 \times (c + 1)$ row vector $(\boldsymbol{\pi}, \pi_{m+1})$, where $\boldsymbol{\pi}$ is an $1 \times c$ row vector. A *discrete phase-type* with representation $(\boldsymbol{\pi}, \mathbf{Q})$, is the distribution of M to reach the absorbing state $c + 1$. Note that the knowledge of $(\boldsymbol{\pi}, \mathbf{Q})$ is sufficient, since $\boldsymbol{u}^T = (\boldsymbol{I}_c - \mathbf{Q})\boldsymbol{e}_c^T$ and $\pi_{c+1} = 1 - \boldsymbol{\pi}\boldsymbol{e}_c^T$, where $\boldsymbol{I}_c$ is the identity matrix of order c and $\boldsymbol{e}_c^T$ is the $c \times 1$ column vector of ones. The matrix $\mathbf{Q}$ must satisfy the condition that $\boldsymbol{I}_c - \mathbf{Q}$ is non-singular. Note that the vector $\boldsymbol{u}^T$ contains the so-called *exit probabilities*, the dimension c of $\mathbf{Q}$ is the called the *order* of the discrete phase-type distribution and the transient states $\{1, 2, \cdots, c\}$, are called the *phases*. Therefore, we shall say that the rv M has a *discrete phase-type distribution of order c* and we shall use the notation $M \sim DPH_c(\boldsymbol{\pi}, \mathbf{Q})$. Then, the probability mass function (pmf) and the distribution function (df) of the rv $M \sim DPH_c(\boldsymbol{\pi}, \mathbf{Q})$, may be expressed respectively as

$$q_m = \Pr(M = m) = \boldsymbol{\pi}\mathbf{Q}^{m-1}\boldsymbol{u}^T, \quad m \in \mathbb{N}_+, \text{ with } \Pr(M = 0) = \pi_{c+1}$$

and

$$\Pr(M \le m) = 1 - \boldsymbol{\pi}\mathbf{Q}^m\boldsymbol{e}_c^T, \quad m \in \mathbb{N}.$$

Hence, using (3), in the following proposition we obtain the distribution of the total number of shocks until the failure of the system by considering that the random change point M has a discrete phase-type distribution. The proof is similar to Theorem 1 of Chadjiconstantinidis and Eryilmaz (2023) and hence omitted. Note that, in the Proposition, the two cases $p_2\boldsymbol{I}_c = p_1\boldsymbol{Q}$ and $p_2\boldsymbol{I}_c \neq p_1\boldsymbol{Q}$ are considered separately since these cases yield different results for the following sum:

$$\sum_{m=1}^{n-1}\left(\frac{p_1}{p_2}\boldsymbol{Q}\right)^{m-1}.$$

Proposition 1. *If X_i have a joint distribution F_1 for $i = 1, 2, \cdots, M$, X_i have a joint distribution F_2 for $i = M + 1, M + 2, \cdots$, and $M \sim DPH_c(\boldsymbol{\pi}, \boldsymbol{Q})$, then the pmf of N_δ is given by*

$$\Pr(N_\delta = n) = (n - 1)(p_1 - p_2)(1 - p_2)p_2^{n-2} + (1 - p_1)p_2^{n-1}, \tag{4}$$

if $p_2\boldsymbol{I}_c = p_1\boldsymbol{Q}$, and

$$\Pr(N_\delta = n) = (1 - p_2)p_1\boldsymbol{\pi}[p_2\boldsymbol{I}_c - p_1\boldsymbol{Q}]^{-1}[p_2^{n-1}\boldsymbol{I}_c - (p_1\boldsymbol{Q})^{n-1}]\boldsymbol{u}^T + (1 - p_1)p_1^{n-1}\boldsymbol{\pi}\boldsymbol{Q}^{n-1}\boldsymbol{e}_c^T, \tag{5}$$

if $p_2\boldsymbol{I}_c \neq p_1\boldsymbol{Q}$.

Remark 2.1. (i) *Let $p_2\boldsymbol{I}_c = p_1\boldsymbol{Q}$.*
Consider the random variables W_1 and W_2, where W_1 has the shifted negative binomial distribution with pmf

$$\Pr(W_1 = n) = (n - 1)(1 - p_2)^2 p_2^{n-2}, \quad n = 2, 3, \cdots$$

and W_2 has the geometric distribution with pmf

$$\Pr(W_2 = n) = (1 - p_2)p_2^{n-1}, \quad n = 1, 2, \cdots$$

and let $\omega = (p_1 - p_2)/(1 - p_2)$. Then, Eq. (4) *can be rewritten as*

$$\Pr(N_\delta = n) = \frac{p_1 - p_2}{1 - p_2}(n - 1)(1 - p_2)^2 p_2^{n-2} + \frac{1 - p_1}{1 - p_2}(1 - p_2)p_2^{n-1},$$

or equivalently

$$\Pr(N_\delta = n) = \omega\, Pr(W_1 = n) + (1 - \omega)\, Pr(W_2 = n), \quad n = 1, 2, \cdots.$$

It should be noted that the weights ω and $1 - \omega$ may take negative values and thus from the above equation it follows that if $p_2\boldsymbol{I}_c = p_1\boldsymbol{Q}$, the distribution of N_δ is a generalized discrete mixture of negative binomial and geometric distributions with weights ω and $1 - \omega$ respectively. We observe that this result holds true for any discrete phase-type distribution M.

(ii) *Let* $p_2\boldsymbol{I}_c \neq p_1\boldsymbol{Q}$.

For simplicity, let us consider that the random variable M *follows a geometric distribution with pmf* $q_m = \Pr(M=m) = p(1-p)^{m-1}$, $m = 1, 2, \cdots$, $0 < p < 1$. *The choice of the model of the geometric distribution is quite meaningful since this can be used as the waiting time for the first occurrence of a certain event. For this special case,* $M \sim DPH_1(\boldsymbol{\pi}, \boldsymbol{Q})$ *with* $\boldsymbol{\pi} = 1$, $\boldsymbol{Q} = 1-p$, $\boldsymbol{u}^T = p$, *and hence the condition* $p_2\boldsymbol{I}_c \neq p_1\boldsymbol{Q}$ *becomes* $p_2 \neq (1-p)p_1$. *Let the random variable* U_1 *has the geometric distribution with pmf*

$$\Pr(U_1 = n) = [1-(1-p)p_1][(1-p)p_1]^{n-1}, \quad n = 1, 2, \cdots.$$

In this case, Eq. (5) *is reduced to*

$$\Pr(N_\delta = n) = \left\{\frac{(1-p_1)pp_1}{(1-p)p_1-p_2} + 1 - p_1\right\}[(1-p)p_1]^{n-1} - \frac{(1-p_2)pp_1}{(1-p)p_1-p_2}p_2^{n-1}$$
$$= \frac{1-p_2}{(1-p)p_1-p_2}[1-(1-p)p_1][(1-p)p_1]^{n-1} - \frac{pp_1}{(1-p)p_1-p_2}(1-p_2)p_2^{n-1},$$

or equivalently

$$\Pr(N_\delta = n) = \theta\, Pr(U_1 = n) + (1-\theta)\, Pr(W_2 = n), \quad n = 1, 2, \cdots$$

where $\theta = (1-p_2)/[(1-p)p_1 - p_2]$. *Hence, the distribution of* N_δ *is a generalized discrete mixture of two geometric distributions with weights* θ *and* $1-\theta$ *respectively.*

According to the proposed δ-shock model, the lifetime of the system is defined by the random sum

$$T_\delta = \sum_{i=1}^{N_\delta} X_i, \tag{6}$$

and hence T_δ is a compound random variable.

Usually, in the study of the compound distributions, it is assumed that the random variables X_i are independent and identically distributed and X_i, N_δ are independent. But in the random sum (6):

(i) The random variables X_i are independent but not identically distributed.

(ii) The random variables X_i and N_δ are dependent.

Therefore, using this representation for T_δ is mathematically intractable to find the distribution of T_δ.

Let X denote a generic random variable of X_i's for $i = 1, 2, \cdots, M$, and $\tilde{X}$ denote a generic random variable of X_i's for $i = M+1, M+2, \cdots$. We suppose that for $i = 1, 2, \cdots, M$, the interarrival times X_i between successive shocks have a continuous distribution with an arbitrary common df $F_1(t) = 1 - \overline{F}_1(t) = \Pr(X \leq t)$ and let $f_1(t) = F_1'(t)$ denotes its common probability density function (pdf). Similarly, for $i = M+1, M+2, \cdots$, the random variables X_i have a continuous distribution with an arbitrary common df $F_2(t) = 1 - \overline{F}_2(t) = \Pr(\tilde{X} \leq t)$ and pdf $f_2(t) = F_2'(t)$. Manifestly, X has df F_1 and $\tilde{X}$ has df F_2. Let $Q_m = \Pr(M > m) = \sum_{n=m+1}^{\infty} q_n$ denote the survival function of the random change point M. A general formula for evaluating the reliability function $\Pr(T_\delta > t)$ of system's lifetime is given in the following Theorem.

Theorem 1. *Let* X_i *have a joint distribution function* F_1 *for* $i = 1, 2, \cdots, M$, *and* X_i *have a joint distribution function* F_2 *for* $i = M+1, M+2, \cdots$, *then*

$$\Pr(T_\delta > t) = \sum_{n=2}^{\infty}\sum_{m=1}^{n-1} p_1^m p_2^{n-m-1} q_m \int_0^\delta \left[P\left(S_{m,1}^* + S_{n-m-1,2}^* > t - x\right)\right] dF_2(x)$$
$$+ \sum_{n=1}^{\infty} p_1^{n-1} Q_{n-1} \int_0^\delta \Pr\left(S_{n-1,1}^* > t - x\right) dF_1(x), \tag{7}$$

where $S_{n,1}^*$ *is the n-th arrival of a renewal process with interarrival times that have the common distribution function*

$$F_{\delta,1}^*(x) = 1 - \frac{\overline{F}_1(x)}{\overline{F}_1(\delta)}, \quad x > \delta$$

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and $S^*_{n,2}$ is the n-th arrival of a renewal process with interarrival times that have the common distribution function

$$F^*_{\delta,2}(x) = 1 - \frac{\overline{F}_2(x)}{\overline{F}_2(\delta)}, \quad x > \delta.$$

Proof. By conditioning on the value of N_δ, from the law of total probability we have

$$\begin{aligned}\Pr(T_\delta > t) &= Pr\{\textstyle\sum_{i=1}^{N_\delta} X_i > t\} \\ &= \textstyle\sum_{n=1}^{\infty} \Pr(S_n > t, N_\delta = n) \\ &= \textstyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Pr(S_n > t, N_\delta = n | M = m) q_m \\ &= \textstyle\sum_{n=1}^{\infty}\sum_{m=1}^{n-1} \Pr(S_n > t, N_\delta = n | M = m) q_m \\ &\quad + \textstyle\sum_{n=1}^{\infty}\sum_{m=n}^{\infty} \Pr(S_n > t, N_\delta = n | M = m) q_m. \end{aligned} \tag{8}$$

• For $1 \le m \le n-1$ we have

$$\begin{aligned}\Pr(S_n > t, N_\delta = n | M = m) = \Pr(&S_n > t, X_1 > \delta, \dots, X_m > \delta, \\ &X_{m+1} > \delta, \dots, X_{n-1} > \delta, X_n \le \delta),\end{aligned}$$

where the independent random variables $\{X_1, \cdots, X_m\}$ have common df F_1, and the independent random variables $\{X_{m+1}, \cdots, X_n\}$ have common df F_2.

Because $S_n = S_{n-1} + X_n$ and X_n is independent of S_{n-1} one obtains

$$\begin{aligned}\Pr(S_n > t, N_\delta = n | M = m) &= \Pr(X_1 > \delta, \dots, X_{n-1} > \delta) \\ &\quad \times \int_0^\delta \Pr(S_{n-1} > t - x | X_1 > \delta, \dots, X_{n-1} > \delta) dPr(X_n \le x) \\ &= [\overline{F}_1(\delta)]^m [\overline{F}_2(\delta)]^{n-m-1} \\ &\quad \times \int_0^\delta \Pr(S_{n-1} > t - x | X_1 > \delta, \dots, X_{n-1} > \delta) dF_2(x). \end{aligned} \tag{9}$$

Note that for $1 \le m \le n-1$,

$$S_{n-1} = \textstyle\sum_{i=1}^{m} Z_i + \sum_{i=m+1}^{n-1} Z_i,$$

where the independent random variables $\{Z_1, \cdots, Z_m\}$ have common df

$$\Pr(Z_i \le x) = 1 - \frac{\overline{F}_1(x)}{\overline{F}_1(\delta)}, \quad x > \delta, \quad 1 \le i \le m$$

and $\{Z_{m+1}, \cdots, Z_{n-1}\}$ have common df

$$\Pr(Z_i \le x) = 1 - \frac{\overline{F}_2(x)}{\overline{F}_2(\delta)}, \quad x > \delta, \quad m+1 \le i \le n-1.$$

Hence, it holds

$$\Pr(S_{n-1} > t - x \big| X_1 > \delta, \ \cdots, X_{n-1} > \delta) = Pr(\textstyle\sum_{i=1}^{m} Z_i + \sum_{i=m+1}^{n-1} Z_i > t - x).$$

Now, from the above it follows that Eq. (9) becomes

$$\Pr(S_n > t, N_\delta = n \big| M = m) = p_1^m p_2^{n-m-1} \int_0^\delta \big[P\big(S^*_{m,1} + S^*_{n-m-1,2} > t - x\big)\big] dF_2(x). \tag{10}$$

• For $m \ge n$ we have

$$\Pr(S_n > t, N_\delta = n | M = m) = \Pr(S_n > t, X_1 > \delta, \dots, X_{n-1} > \delta, X_n \le \delta\},$$

where now the independent random variables $\{X_1, \cdots, X_n\}$ have common df F_1. Hence, from the proof of Lemma 1 in Eryilmaz and Bayramoglu (2014) we get that

$$\begin{aligned}&\Pr(S_n > t, X_1 > \delta, \dots, X_{n-1} > \delta, X_n \le \delta) \\ &\quad = \Pr(X_1 > \delta, \dots, X_{n-1} > \delta) \\ &\quad \quad \times \int_0^\delta \Pr(S_{n-1} > t - x | X_1 > \delta, \dots, X_{n-1} > \delta) dPr(X_n \le x) \\ &\quad = [\overline{F}_1(\delta)]^{n-1} \int_0^\delta \Pr(S_{n-1} > t - x | X_1 > \delta, \dots, X_{n-1} > \delta) dF_1(x) \\ &\quad = [\overline{F}_1(\delta)]^{n-1} \int_0^\delta \Pr\big(S^*_{n-1,1} > t - x\big)\, dF_1(x),\end{aligned}$$

and thus

$$\begin{aligned}&\textstyle\sum_{n=1}^{\infty}\sum_{m=n}^{\infty} \Pr(S_n > t, N_\delta = n | M = m) q_m \\ &\quad = \textstyle\sum_{n=1}^{\infty}\sum_{m=n}^{\infty} [\overline{F}_1(\delta)]^{n-1} \int_0^\delta \Pr\big(S^*_{n-1,1} > t - x\big)\, dF_1(x)\, q_m \\ &\quad = \textstyle\sum_{n=1}^{\infty} [\overline{F}_1(\delta)]^{n-1} \int_0^\delta \Pr\big(S^*_{n-1,1} > t - x\big)\, dF_1(x) \sum_{m=n}^{\infty} q_m \\ &\quad = \textstyle\sum_{n=1}^{\infty} p_1^{n-1} Q_{n-1} \int_0^\delta \Pr\big(S^*_{n-1,1} > t - x\big)\, dF_1(x). \end{aligned} \tag{11}$$

Substituting (10) and (11) into the right-hand side of (8), we get immediately (7). ■

In the sequel, we give another approach to study the reliability of system's lifetime, by deriving the Laplace-Stieltjes transform (LST) of T_δ. Define the Laplace-Stieltjes transforms (LST) of the conditional random variables $X|X > \delta$ and $X|X \le \delta$ by

$$\hat{\xi}_1(u) = E(e^{-uX}|X > \delta), \quad \hat{\zeta}_1(u) = E(e^{-uX}|X \le \delta) \tag{12}$$

and the LSTs of the conditional random variables $\tilde{X}|\tilde{X} > \delta$ and $\tilde{X}|\tilde{X} \le \delta$ by

$$\hat{\xi}_2(u) = E\left(e^{-u\tilde{X}}\middle|\tilde{X} > \delta\right), \quad \hat{\zeta}_2(u) = E\left(e^{-u\tilde{X}}\middle|\tilde{X} \le \delta\right). \tag{13}$$

Then, we have the following.

Theorem 2. *The Laplace-Stieltjes transform $\hat{f}_\delta(u) = E[e^{-uT_\delta}]$ of T_δ, is given by*

$$\hat{f}_\delta(u) = \frac{(1-p_1)\hat{\zeta}_1(u)}{1-p_1\hat{\xi}_1(u)} + \left\{\frac{(1-p_2)\hat{\zeta}_2(u)}{1-p_2\hat{\xi}_2(u)} - \frac{(1-p_1)\hat{\zeta}_1(u)}{1-p_1\hat{\xi}_1(u)}\right\}\sum_{m=1}^{\infty} p_1^m \hat{\xi}_1^m(u) q_m \tag{14}$$

where $\hat{\xi}_i(u)$ and $\hat{\zeta}_i(u)$, $i = 1, 2$, are given by (13) *and* (14).

Proof. For a fixed m it holds

$$E[e^{-uT_\delta}|M = m] = \sum_{n=1}^{\infty} E[e^{-uT_\delta}|(N_\delta = n|M = m)]\Pr(N_\delta = n|M = m),$$

and hence

$$\begin{aligned} E[e^{-uT_\delta}] &= \sum_{m=1}^{\infty} E[e^{-uT_\delta}|M = m]q_m \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E[e^{-uT_\delta}|(N_\delta = n|M = m)]\Pr(N_\delta = n|M = m)q_m \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{m} E[e^{-uT_\delta}|(N_\delta = n|M = m)]\Pr(N_\delta = n|M = m)q_m \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty} E[e^{-uT_\delta}|(N_\delta = n|M = m)]\Pr(N_\delta = n|M = m)q_m. \end{aligned}$$

By the definition of the random variable N_δ, we get

$$\begin{aligned} \hat{f}_\delta(u) &= \sum_{m=1}^{\infty}\sum_{n=1}^{m} E\left[e^{-u\sum_{i=1}^{N_\delta} X_i}|X_1 > \delta, \dots, X_{n-1} > \delta,\, X_n \le \delta\right](1-p_1)p_1^{n-1}\, q_m \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty} E\left[e^{-u\sum_{i=1}^{N_\delta} X_i}|X_1 > \delta, \dots, X_m > \delta,\; X_{m+1} > \delta, \dots, X_{n-1} > \delta,\, X_n \le \delta\right] \\ &\quad \times (1-p_2)p_1^m p_2^{n-m-1} q_m \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{m} \hat{\xi}_1^{n-1}(u)\, \hat{\zeta}_1(u)\, (1-p_1)p_1^{n-1}\, q_m \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty} \hat{\xi}_1^m(u)\, \hat{\xi}_2^{n-m-1}(u)\hat{\zeta}_2(u)\, (1-p_2)p_1^m p_2^{n-m-1} q_m, \end{aligned}$$

from which (14) follows easily. ■

Using that $E[T_\delta] = -\hat{f}_\delta(0)$, from Theorem 2, we directly obtain the mean time to failure (MTTF) of the system. Hence, we have the following

Corollary 1. *The mean time to failure (MTTF) of the system is*

$$\begin{aligned} E[T_\delta] &= \left\{E[X|X \le \delta] + \frac{p_1}{1-p_1} E[X|X > \delta]\right\}\left\{1 - \sum_{m=1}^{\infty} p_1^m q_m\right\} \\ &\quad + \left\{E[\tilde{X}\,|\tilde{X} \le \delta] + \frac{p_2}{1-p_2} E[\check{X}|\tilde{X} > \delta]\right\}\sum_{m=1}^{\infty} p_1^m q_m. \end{aligned}$$

If the random variable M follows a discrete phase-type distribution, from Theorem 2 we get the following

Corollary 2. *Let $M \sim PH_c(\boldsymbol{\pi}, \boldsymbol{Q})$. Then, the Laplace-Stieltjes transform $\hat{f}_\delta(u) = E[e^{-uT_\delta}]$ of T_δ, is given by*

$$\hat{f}_\delta(u) = \frac{(1-p_1)\hat{\zeta}_1(u)}{1-p_1\hat{\xi}_1(u)} + \left\{\frac{(1-p_2)\hat{\zeta}_2(u)}{1-p_2\hat{\xi}_2(u)} - \frac{(1-p_1)\hat{\zeta}_1(u)}{1-p_1\hat{\xi}_1(u)}\right\} p_1\hat{\xi}_1(u)\boldsymbol{\pi}\left(\boldsymbol{I}_c - p_1\hat{\xi}_1(u)\boldsymbol{Q}\right)^{-1}\boldsymbol{u}^T,$$

where $\hat{\xi}_i(u)$ and $\hat{\zeta}_i(u)$, $i = 1, 2$, are given by (12) *and* (13).

Proof. Since

$$\begin{aligned} \sum_{m=1}^{\infty} p_1^m\, \hat{\xi}_1^m(u) q_m &= \sum_{m=1}^{\infty} p_1^m\, \hat{\xi}_1^m(u)\boldsymbol{\pi}\mathbf{Q}^{m-1}\boldsymbol{u}^T \\ &= p_1\hat{\xi}_1(u)\boldsymbol{\pi}\left\{\sum_{m=1}^{\infty}\left[p_1\hat{\xi}_1(u)\boldsymbol{Q}\right]^{m-1}\right\}\boldsymbol{u}^T \end{aligned}$$

$$= p_1\hat{\xi}_1(u)\boldsymbol{\pi}\big(\boldsymbol{I}_c - p_1\hat{\xi}_1(u)\boldsymbol{Q}\big)^{-1}\boldsymbol{u}^T,$$

the result follows from (14). ■

Also, using Corollary 1, we immediately obtain the following

Corollary 3. *Let $M \sim PH_c(\boldsymbol{\pi}, \boldsymbol{Q})$. Then, the mean time to failure (MTTF) of the system is*

$$E[T_\delta] = \left\{E[X|X \le \delta] + \frac{p_1}{1-p_1}E[X|X > \delta]\right\}\{1 - p_1\boldsymbol{\pi}(\boldsymbol{I}_c - p_1\boldsymbol{Q})^{-1}\boldsymbol{u}^T\}$$
$$+p_1\left\{E[\tilde{X}\,\middle|\tilde{X} \le \delta] + \frac{p_2}{1-p_2}E[\check{X}|\tilde{X} > \delta]\right\}\boldsymbol{\pi}(\boldsymbol{I}_c - p_1\boldsymbol{Q})^{-1}\boldsymbol{u}^T.$$

Now, let us consider some specific discrete phase-type distributions for the random variable M. At first, we consider that M has the geometric distribution with probability mass function (pmf) $q_m = \Pr(M = m) = p(1-p)^{m-1}$, $m = 1, 2, \cdots, 0 < p < 1$.

Corollary 4. *Let $q_m = p(1-p)^{m-1}$, $m = 1,\ 2,\ \cdots,\ 0 < p < 1$. Then, the Laplace-Stieltjes transform $\hat{f}_\delta(u) = E[e^{-uT_\delta}]$ of T_δ, is given by*

$$\hat{f}_\delta(u) = \frac{(1-p_1)\hat{\zeta}_1(u)}{1-qp_1\hat{\xi}_1(u)} + \frac{(1-p_2)pp_1\hat{\xi}_1(u)\hat{\zeta}_2(u)}{[1-qp_1\hat{\xi}_1(u)][1-p_2\hat{\xi}_2(u)]}, \tag{15}$$

where $\hat{\xi}_i(u)$ and $\hat{\zeta}_i(u)$, $i = 1, 2$, are given by (16) *and* (17), $q = 1 - p$.

Proof. For $q_m = p(1-p)^{m-1}$, $m \ge 1$, it holds that $M \sim PH_c(\boldsymbol{\pi}, \boldsymbol{Q})$ with $c = 1$, $\boldsymbol{\pi} = 1$, $\boldsymbol{Q} = 1 - p$ and $\boldsymbol{u}^T = p$. Hence, (15) follows from Corollary 2. ■

Also, using Corollary 3, we immediately obtain the following

Corollary 5. *Let $q_m = p(1-p)^{m-1}$, $m = 1, 2, \cdots, 0 < p < 1$. Then, the mean time to failure (MTTF) of the system is*

$$E[T_\delta] = \frac{1-p_1}{1-(1-p)p_1}\left\{E[X|X \le \delta] + \frac{p_1}{1-p_1}E[X|X > \delta]\right\}$$
$$+\frac{pp_1}{1-(1-p)p_1}\left\{E[\tilde{X}\,\middle|\tilde{X} \le \delta] + \frac{p_2}{1-p_2}E[\check{X}|\tilde{X} > \delta]\right\}.$$

Now, we consider that the random variable M follows the negative binomial distribution with pmf $q_m = \Pr(M = m) = (m-1)p^2(1-p)^{m-2}$, $m = 2, 3, \cdots, 0 < p < 1$. Then, we have the following.

Corollary 6. *Let $q_m = (m-1)p^2(1-p)^{m-2}$, $m = 2,\ 3,\ \cdots,\ 0 < p < 1$. Then, the Laplace-Stieltjes transform $\hat{f}_\delta(u) = E[e^{-uT_\delta}]$ of T_δ, is given by*

$$\hat{f}_\delta(u) = \frac{(1-p_1)\hat{\zeta}_1(u)}{1-p_1\hat{\xi}_1(u)} + \left\{\frac{(1-p_2)\hat{\zeta}_2(u)}{1-p_2\hat{\xi}_2(u)} - \frac{(1-p_1)\hat{\zeta}_1(u)}{1-p_1\hat{\xi}_1(u)}\right\}\left(\frac{pp_1\hat{\xi}_1(u)}{1-(1-p)p_1\hat{\xi}_1(u)}\right)^2. \tag{16}$$

Proof. For $q_m = (m-1)p^2(1-p)^{m-2}$, $m \ge 2$, it holds that $M \sim PH_c(\boldsymbol{\pi}, \boldsymbol{Q})$ with

$$c = 2, \boldsymbol{\pi} = (1, 0),\ \ \boldsymbol{Q} = \begin{bmatrix} 1-p & p \\ 0 & 1-p \end{bmatrix} \text{ and } \boldsymbol{u} = (0, p),$$

and thus (16) follows from Corollary 2. ■

Similarly, using Corollary 1, we get the following.

Corollary 7. *Let $q_m = (m-1)p^2(1-p)^{m-2}$, $m = 2, 3, \cdots, 0 < p < 1$. Then, the MTTF of the system is*

$$E[T_\delta] = \frac{(1-p_1)[1-(1-2p)p_1]}{[1-(1-p)p_1]^2}\left\{E[X|X \le \delta] + \frac{p_1}{1-p_1}E[X|X > \delta]\right\}$$
$$+\left(\frac{pp_1}{1-(1-p)p_1}\right)^2\left\{E[\tilde{X}\,\middle|\tilde{X} \le \delta] + \frac{p_2}{1-p_2}E[\check{X}|\tilde{X} > \delta]\right\}.$$

In Corollaries 4-7, geometric and negative Binomial distributions have been considered to model the distribution of the change point. These well-known classical waiting time distributions are suitable for modeling the occurrence time of a specific event. Indeed, in our case, the change in the law of the intershock time may occur with respect to another extreme event whose distribution can be modeled by waiting time distributions such as geometric and negative Binomial. Since the class of discrete phase-type distributions involves geometric and negative Binomial distributions and some of their generalizations, we have presented general results in Corollaries 2-3.

3. Numerical example

For an illustration, let us consider that the interarrival times between successive shocks follow exponential distributions with pdfs $f_i(t) = \lambda_i e^{-\lambda_i t}, \quad t \geq 0, \lambda_i > 0, i = 1, 2.$
Then,

$$E[X|X > \delta] = \delta + \frac{1}{\lambda_1}, \; E[X|X \leq \delta] = \frac{\frac{1}{\lambda_1} - \left[\delta + \frac{1}{\lambda_1}\right] e^{-\lambda_1 \delta}}{1 - e^{-\lambda_1 \delta}},$$

and

$$E[\breve{X}|\tilde{X} > \delta] = \delta + \frac{1}{\lambda_2}, \; E[\tilde{X}|\tilde{X} \leq \delta] = \frac{\frac{1}{\lambda_2} - \left[\delta + \frac{1}{\lambda_2}\right] e^{-\lambda_2 \delta}}{1 - e^{-\lambda_2 \delta}}.$$

In Fig. 1, we plot the MTTF of the system when $q_m = p(1-p)^{m-1}, m = 1, 2, \cdots$, as a function of the parameter p for selected values of δ when $\lambda_1 = 0.1, \lambda_2 = 0.2$. An inrease in p leads to an earlier change point and hence more frequent shocks (for selected values of the parameters λ_1 and λ_2) occur earlier. Such a behavior leads to a decrease in MTTF of the system. As expected , the MTTF is a decreasing function of δ. With an increase in δ, the magnitude of the change in the MTTF with respect to the value of the parameter p is less. Indeed, as observed from Figure 1, the change in MTTF is sharper for small values of the parameter p when $\delta = 1$. Figure 2 plots the MTTF of the system as a function of δ for selected values of p.

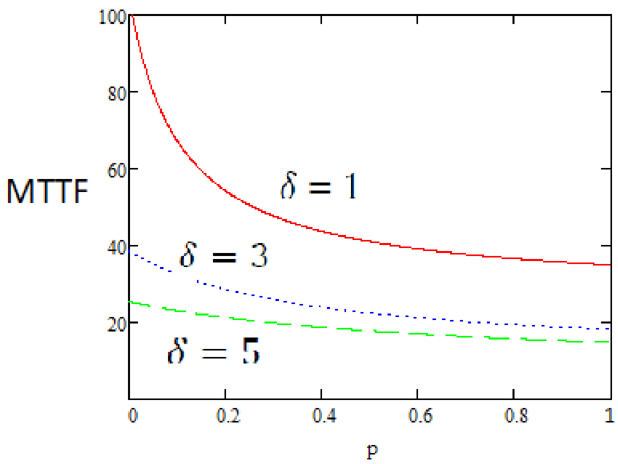

Figure 1. The MTTF of the system as a function of p for selected values of δ.

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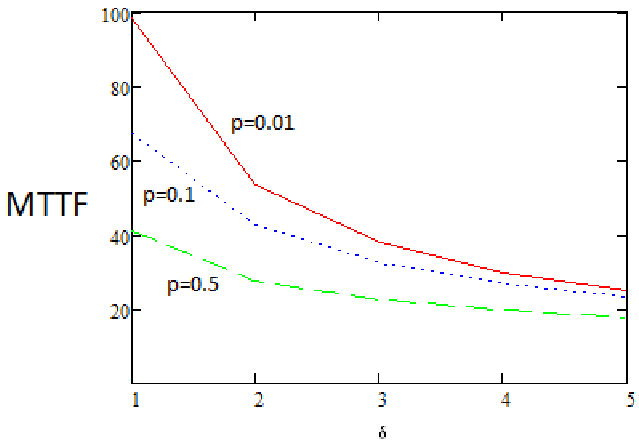

Figure 2. The MTTF of the system as a function of δ for selected values of p.

Under the same assumptions, using Theorem 1, the following expression can be obtained for the survival function of the system:

$$\Pr(T_\delta > t) = \sum_{n=2}^{\infty}\sum_{m=1}^{n-1}\left(e^{-\lambda_1\delta}\right)^m\left(e^{-\lambda_2\delta}\right)^{n-m-1}p(1-p)^{m-1}$$
$$\times\int_0^\delta\int_0^\infty P\left(S_{m,1}^* > t-x-y\right)f_{S_{n-m-1,2}^*}(y)\lambda_2 e^{-\lambda_2 x}dydx$$
$$+\sum_{n=1}^{\infty}\left(e^{-\lambda_1\delta}\right)^{n-1}(1-p)^{n-1}\int_0^\delta \Pr\left(S_{n-1,1}^* > t-x\right)\lambda_1 e^{-\lambda_1 x}\,dx,$$

where $S_{m,1}^*$ and $S_{n-m-1,2}^*$ respectively are the sums of m and $n-m-1$ independent and identical random variables having common cdf

$$F_{\delta,1}^*(x) = 1-\frac{\bar{F}_1(x)}{\bar{F}_1(\delta)} = 1-e^{-\lambda_1(x-\delta)}, \qquad x > \delta,$$

and

$$F_{\delta,2}^*(x) = 1-\frac{\bar{F}_2(x)}{\bar{F}_2(\delta)} = 1-e^{-\lambda_2(x-\delta)}, \qquad x > \delta.$$

It is known that (see, e.g. Cinlar (1975))

$$P\left(S_{m,1}^* > t\right) = \begin{cases} 1, & if\ t < m\delta \\ \sum_{k=0}^{m-1} e^{-\lambda_1(t-m\delta)}\dfrac{(t-m\delta)^k}{k!}, & if\ t \ge m\delta \end{cases}$$

4. Summary and Conclusions

In this paper, the well-known δ-shock model has been studied when there is a change in the distribution of the intershock times with respect to an arbitrary waiting time random variable. The survival function and MTTF of the system have been derived under this change point setup. The LST of the system's lifetime has also been obtained.

In the developments, only one change point has been considered. This may be suitable if a system or unit experiences only one sudden change in its operating environment during its lifetime. However, in practice, the change point may occur recurrently and after each change another counting process can be involved. This general setting is probabilistically more difficult to work especially to obtain the survival function of the system. However, some satisfactory approximations and/or bounds may be obtained for the survival function when there are more than one change point. This will be among our future research problems.

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